Controlled approximation and interpolation for some classes of holomorphic functions

Mark Elin

Department of Mathematics, ORT Braude College,

P.O. Box 78, Karmiel 21982, ISRAEL

e-mail: mark.elin@gmail.com

David Shoikhet

Department of Mathematics, ORT Braude College,

P.O. Box 78, Karmiel 21982, ISRAEL

e-mail: davs27@netvision.net.il

Lawrence Zalcman*

Department of Mathematics, Bar-Ilan University,

52900 Ramat-Gan, ISRAEL

e-mail: zalcman@macs.biu.ac.il

Abstract

This paper reports on constructive approximation methods for three classes of holomorphic functions on the unit disk which are closely connected each other: the class of starlike and spirallike functions, the class of semigroup generators, and the class of functions with positive real part. It is more-or-less known that starlike or spirallike functions can be defined as solutions of singular differential equations which, in general, are not stable under the motion of interior singular points to the boundary. At the same time, one can establish a perturbation formula which continuously transforms a starlike (or spirallike) function with respect to a boundary point to a starlike (or spirallike)

*Research supported by the German–Israel Foundation for Scientific Research and Development, G.I.F. Grant No. I-809-234-6/2003.

function with respect to an interior point. This formula is based on an appropriate approximation method of holomorphic generators which determine the above-mentioned differential equations. In turn, the well-known Berkson–Porta parametric representation of holomorphic generators leads us to study an approximation-interpolation problem for the class of functions with positive real part. While this problem is of independent interest, the solution we present here is again based on the Berkson–Porta formula. Finally, we apply our results to solve a natural perturbation problem for one-parameter semigroups of holomorphic self-mappings, as well as the eigenvalue problem for the semigroup of composition operators.

1 Preliminaries

1.1 Starlike and spirallike functions

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and $\Pi = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ the right half-plane. We denote the set of holomorphic functions on Δ which take values in a set $\Omega \subset \mathbb{C}$ by $\mathrm{Hol}(\Delta, \Omega)$. In particular, $\mathrm{Hol}(\Delta, \mathbb{C})$ is the set of all holomorphic functions on Δ. This set is a Frechét space endowed with the seminorms $\|f\|_K := \max_{z\in K} |f(z)|$, where K is a compact subset of Δ.

For brevity, we write $\mathrm{Hol}(\Delta)$ for $\mathrm{Hol}(\Delta, \Delta)$ and $\mathcal{P}$ for $\mathrm{Hol}(\Delta, \Pi)$. The set $\mathrm{Hol}(\Delta)$ is a semigroup with respect to composition.

We denote the subset of $\mathrm{Hol}(\Delta, \mathbb{C})$ consisting of univalent functions (without any normalization) by $\mathrm{Univ}(\Delta)$.

Definition 1.1 *A univalent function h is said to be* starlike *if for each point $z \in \Delta$, the linear segment joining $h(z)$ to zero*

$$(0, h(z)] := \{th(z) : \ t \in (0, 1]\}$$

lies entirely in the image $h(\Delta)$.

Definition 1.2 *A univalent function h is said to be* spirallike *if there is a number $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > 0$ such that for every point $z \in \Delta$, the spiral curve*

$$\{e^{-t\mu}h(z) : \ t \geq 0\}$$

lies entirely in $h(\Delta)$.

It is clear that if μ can be chosen to be real, then the spirallike function h is actually starlike.

It follows from the above definitions that $0 \in \overline{h(\Delta)}$ for each starlike or spirallike function h. We distinguish two situations.

• In case $0 \in h(\Delta)$, the starlike (spirallike) function h is said to be starlike (spirallike) with respect to an interior point.

• In case $0 \in \partial h(\Delta)$, the function h is said to be starlike (spirallike) with respect to a boundary point.

For any starlike (spirallike) function with respect to an interior point, there exists a unique point $\tau \in \Delta$ such that $h(\tau) = 0$.

If h is starlike (spirallike) with respect to a boundary point, then one can show (see, for example, [11, 12]) that there exists a unique point $\tau \in \partial\Delta$ such that the value $h(\tau)$ defined as the angular limit at this point is equal to zero: $h(\tau) := \angle \lim\limits_{z\to\tau} h(z) = 0$.

In both cases, we use the notation $S^*[\tau]$ for the set of starlike functions (with respect to an interior or a boundary point) satisfying the condition $h(\tau) = 0$, $\tau \in \overline{\Delta}$, and the notation $\mathrm{Sp}[\tau]$ for the set of spirallike functions satisfying the same condition.

In this paper, we use autonomic dynamical systems to study approximation problems for starlike and spirallike functions with respect to a boundary point and for some related classes of functions. Since such systems are time-independent, their solutions form one-parameter semigroups of holomorphic self-mappings of the open unit disk.

1.2 Semigroups of holomorphic self-mappings

Definition 1.3 *A family $\mathcal{S} = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ of holomorphic self-mappings of Δ is called a* one-parameter continuous semigroup *(respectively,* group*) if*

(i) $F_{t+s} = F_t \circ F_s$ whenever s, t and $s+t$ belong to $\mathbb{R}^+$ (respectively, $\mathbb{R}$);

(ii) $F_0(z) = z$ for all $z \in \Delta$, that is, F_0 is the identity mapping on Δ.

(iii) $\lim\limits_{t\to s} F_t(z) = F_s(z)$ for all $t > 0$, $s \geq 0$ (respectively, $t, s \in \mathbb{R}$) and for all $z \in \Delta$.

The following result is due to Berkson and Porta [6] (see also [21, 23, 1]).

Theorem 1.1 (see Propositions 3.2.1 and 3.2.2 [27]) *Let $\mathcal{S} = \{F_t\}_{t\geq 0}$ be a one-parameter semigroup of holomorphic self-mappings of Δ such that for*

each $z \in \Delta$,

$$\lim_{t \to 0^+} F_t(z) = z. \tag{1.1}$$

Then for each $z \in \Delta$*, the limit*

$$\lim_{t \to 0^+} \frac{z - F_t(z)}{t} = f(z), \tag{1.2}$$

exists and is a holomorphic function on Δ*. The convergence in (1.2) is uniform on each subset strictly inside* Δ*. Moreover, the semigroup* $\mathcal{S}$ *can be defined as the (unique) solution of the Cauchy problem*

$$\begin{cases} \dfrac{\partial F_t(z)}{\partial t} + f(F_t(z)) = 0, \quad t \geq 0, \\ F_0(z) = z, \quad z \in \Delta. \end{cases} \tag{1.3}$$

Definition 1.4 *Let* $\mathcal{S} = \{F_t\}_{t \geq 0}$ *be a one-parameter continuous semigroup of holomorphic self-mappings of* Δ*. The function* $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ *defined by the limit (1.2) is called the* (infinitesimal) generator *of* $\mathcal{S}$*.*

We denote the family of all holomorphic generators on Δ by $\mathcal{G}$. This set is a real cone in $\mathrm{Hol}(\Delta, \mathbb{C})$ [21, 27].

A continuous version of the Denjoy–Wolff Theorem (see [22]) asserts that *if a semigroup* $\mathcal{S} = \{F_t\}_{t \geq 0}$ *does not contain an elliptic automorphism, then there is a unique point* $\tau \in \overline{\Delta}$ *such that* $\lim_{t \to \infty} F_t(z) = \tau$ *for all* $z \in \Delta$*.*

This point τ is called the Denjoy–Wolff point of the semigroup $\mathcal{S}$. For a given $\tau \in \overline{\Delta}$, the set of all semigroups for which τ is their Denjoy–Wolff point is denoted by $\mathcal{B}[\tau]$. The set of functions $f \in \mathcal{G}$ such that the semigroup generated by f belongs to $\mathcal{B}[\tau]$ is denoted by $\mathcal{G}[\tau]$. Note that $\mathcal{G}[\tau]$ is a real subcone of $\mathcal{G}$.

• If $\mathcal{S} = \{F_t\}_{t \geq 0} \in \mathcal{B}[\tau]$, where $\tau \in \Delta$ is an interior point of Δ, then τ must be the unique (interior) fixed point of the semigroup $\mathcal{S}$, i.e., $F_t(\tau) = \tau$ for all $t \geq 0$.

Also, it follows by the uniqueness of the solution of the Cauchy problem (1.3) that, in this case, τ is the unique zero of f in Δ. Moreover, it can be shown (see, for example, Theorem 1.2 below) that $f \in \mathcal{G}$ belongs to $\mathcal{G}[\tau]$, $\tau \in \Delta$, if and only if $f(\tau) = 0$ and $\mathrm{Re}\, f'(\tau) > 0$.

• If $\mathcal{S} = \{F_t\}_{t\geq 0} \in \mathcal{B}[\tau]$, where $\tau \in \partial\Delta$ is a boundary point, then its generator f does not vanish inside Δ, but

$$\angle \lim_{z\to\tau} f(z) = 0 \tag{1.4}$$

and

$$\angle \lim_{z\to\tau} F_t(z) = \tau$$

(see [14]), where the symbol $\angle \lim$ denotes the so-called angular (or non-tangential) limit at a boundary point of Δ (see, for example, [20]).

Note, however, that $f \in \mathcal{G}$ may have more than one boundary null point in the sense of (1.4). Moreover, in general, condition (1.4) does not even imply that τ is a fixed point for the semigroup $\mathcal{S} = \{F_t\}_{t\geq 0}$ generated by f (consider, for example, $f \in \mathcal{G}$ defined by $f(z) = z\sqrt{1-z}$ with $\tau = 1$).

A characterization of the classes $\mathcal{G}[\tau]$, as well as the whole class $\mathcal{G}$, is given in the following assertion.

Theorem 1.2 (see [6, 2, 27]) *The following statements are equivalent:*

(i) $f \in \mathcal{G}$*;*

(ii) $f(z) = a - \bar{a}z^2 + zp(z)$ *for some* $a \in \mathbb{C}$ *and* $p \in \mathcal{P}$*;*

(iii) f *admits the representation*

$$f(z) = (z-\tau)(1-z\bar{\tau})p(z) \tag{1.5}$$

for some $\tau \in \overline{\Delta}$ *and* $p \in \mathcal{P}$*.*

Moreover, for a given $\tau \in \overline{\Delta}$*, every generator* $f \in \mathcal{G}[\tau]$ *admits the representation (1.5). Conversely, if* f *is represented by (1.5) and does not generate a group of elliptic automorphisms, then it belongs to* $\mathcal{G}[\tau]$*.*

The parametric representation (1.5) is due to Berkson and Porta [6]. This representation is unique for $\tau \in \overline{\Delta}$ and $p \in \mathcal{P}$.

The next theorem characterizes the class $\mathcal{G}[\tau]$ specifically for the case where $\tau \in \partial\Delta$ is a boundary Denjoy–Wolff point of generated semigroups.

Theorem 1.3 (see [14]) *Let* $f \in \mathcal{G}$ *be a semigroup generator on* Δ *and let* $\mathcal{S} = \{F_t\}_{t\geq 0}$ *be the semigroup generated by* f*. The following are equivalent.*

(i) f *has no null point in* Δ*.*

(ii) There is a point $\tau \in \partial\Delta$ *such that*

$$\beta = \angle \lim_{z\to\tau} \frac{f(z)}{z-\tau} =: \angle f'(\tau)$$

exists finitely and $\operatorname{Re}\beta \geq 0$*. Consequently,* $f(\tau) = 0$*.*

(iii) There exists a point $\tau \in \partial\Delta$ *and a real nonnegative number* γ *such that*

$$\frac{|F_t(z) - \tau|^2}{1 - |F_t(z)|^2} \leq \exp(-t\gamma) \frac{|z - \tau|^2}{1 - |z|^2}$$

for all $z \in \Delta$*. Consequently,* $f \in \mathcal{G}[\tau]$*.*

Moreover,

(a) the boundary points τ *in (ii)–(iii) are the same;*

(b) the limit β *in (ii) is a real nonnegative number;*

(c) the maximal number $\gamma \geq 0$ *for which (iii) holds coincides with the limit value* β *in (ii).*

This theorem can be considered the infinitesimal version of the Julia–Wolff–Carathéodory Theorem. Assertion (iii) means that all horocycles internally tangent to the boundary $\partial\Delta$ at the point τ are invariant under the semigroup action[1]. Moreover, if $\beta = f'(\tau) > 0$, then choosing $\gamma = \beta$, we get an exponential rate of convergence of the semigroup $\mathcal{S} = \{F_t\}_{t\geq 0}$ to τ. Since we are mostly interested in such a situation, we denote by $\mathcal{G}^+[\tau]$ ($\tau \in \partial\Delta$) the set of all generators satisfying the conditions

$$\angle\lim_{z\to\tau} f(z) = 0, \quad \angle\lim_{z\to\tau} \frac{f(z)}{z-\tau} = \beta > 0.$$

1.3 Differential and functional equations for starlike and spirallike functions

The dynamic approach to the study of starlike and spirallike functions is based on the following observation. Let $h \in \operatorname{Univ}(\Delta)$; then h is spirallike (starlike) if and only if there exists $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > 0$ ($\mu \in (0, \infty)$) such that for each $t \geq 0$, the function

$$F_t(z) = h^{-1}\left(e^{-\mu t} h(z)\right) \tag{1.6}$$

is a well-defined holomorphic self-mapping of the open unit disk Δ.

Obviously, the family $\mathcal{S} = \{F_t\}_{t\geq 0} \subset \operatorname{Hol}(\Delta)$ forms a one-parameter

[1] In this case, the point $\tau \in \partial\Delta$ is sometimes also called the sink point of the semigroup.

continuous semigroup. Note also that $h \in \mathrm{Sp}[\tau]$ (or $h \in S^*[\tau]$) if and only if $\mathcal{S} \in \mathcal{B}[\tau]$.

Now, differentiating (1.6) at $t = 0^+$, we see that the function h satisfies the differential equation

$$\mu h(z) = h'(z) f(z), \tag{1.7}$$

where $f = \lim\limits_{t \to 0^+} \frac{z - F_t(z)}{t}$ is the generator of the semigroup $\mathcal{S} = \{F_t\}_{t \geq 0}$.

In fact, the converse assertion also holds. However, as there are some differences between the interior or the boundary location of the the point τ, we describe these cases separately.

Theorem 1.4 *Let $\tau \in \Delta$. A function $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ belongs to $\mathrm{Sp}[\tau]$ if and only if it is locally univalent and satisfies equation (1.7) for some $f \in \mathcal{G}[\tau]$ and $\mu = f'(\tau)$. Moreover, in this case, h also satisfies equation (1.6), where $\mathcal{S} = \{F_t\}_{t \geq 0}$ is the semigroup generated by f.*

Theorem 1.5 *Let $\tau \in \partial\Delta$. A univalent function $h \in \mathrm{Univ}(\Delta)$ belongs to $\mathrm{Sp}[\tau]$ if and only if it satisfies equation (1.7) for some $f \in \mathcal{G}^+[\tau]$ and $\mu \in \mathbb{C}$ with $\mathrm{Re}\,\mu > 0$.*

Remark 1. Note that the set $\mathcal{G}$ of holomorphic generators is a real cone [22, 27]. Hence, for each spirallike (starlike) function h, the pair (μ, f) in equation (1.7) can be replaced by a pair $(\alpha\mu, \alpha f)$ for any $\alpha > 0$. In fact, a function $h \in \mathrm{Univ}(\Delta)$ satisfying equation (1.7) with some $\mu \in \mathbb{C}$, $\mathrm{Re}\,\mu > 0$, and $f \in \mathcal{G}$ is actually starlike if and only if $\frac{1}{\mu} \cdot f \in \mathcal{G}$.

Remark 2. Note that for $\tau \in \Delta$, equation (1.7) has no holomorphic solution when $\mu \neq f'(\tau)$, since this equation has an interior singular point. In other words, the number $\mu \in \mathbb{C}$ is uniquely determined by $f \in \mathcal{G}[\tau]$.

For $\tau \in \partial\Delta$, equation (1.7) has a holomorphic solution for each $\mu \in \mathbb{C}$. However, it can be shown (see Theorem 3.8 below) that this solution is univalent if and only if μ lies in the set $\Omega = \Omega_+ \bigcup \Omega_-$, where $\Omega_\pm = \left\{ w \in \mathbb{C} : \ |w \mp \beta| \leq \beta, w \neq 0 \right\}$, $\beta = f'(\tau) > 0$. Moreover, the solution h of (1.7) normalized by the condition $h(0) = 1$ belongs to $\mathrm{Sp}[\tau]$ (respectively, to $S^*[\tau]$) if and only if $\mu \in \Omega_+$ (respectively, $0 < \mu \leq 2\beta$).

Remark 3. Both Theorem 1.4 and 1.5 state that for a given semigroup $\mathcal{S} = \{F_t\}_{t \geq 0}$ generated by $f \in \mathcal{G}[\tau]$, $\tau \in \overline{\Delta}$, the solution h of the differential

equation (1.7) is a solution of the so-called Schröder equation

$$h(F_t(z)) = \lambda_t h(z) \tag{1.8}$$

with $\lambda_t = e^{-\mu t}$ [9].

On the other hand, if we do not require the univalence of solutions, equation (1.8) can be considered the eigenvalue problem for the semigroup of composition operators $\{C_t\}_{t\geq 0}$ defined by

$$C_t(g)\Big(= C_{F_t}(g)\Big) := g(F_t) \quad \text{for all } g \in \mathrm{Hol}(\Delta, \mathbb{C}).$$

For $\tau \in \Delta$, the spectrum $\sigma(C_{F_t})$ is discrete, while for $\tau \in \partial\Delta$, the spectrum $\sigma(C_{F_t})$ covers the whole region (see the discussion in Section 3 below).

In general, to each $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ there corresponds a vector field Γ_f defined by

$$\Gamma_f(g)(z) := g'(z)f(z) \quad \text{for all } g \in \mathrm{Hol}(\Delta, \mathbb{C}). \tag{1.9}$$

Each vector field Γ_f is locally integrable in the sense that for each $z \in \Delta$, there exists a neighborhood U of z and a number $\delta > 0$ such that the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,z)}{\partial t} + f(u(t,z)) = 0 \\ u(0,z) = z \end{cases} \tag{1.10}$$

has a unique solution $\{u(t,z)\} \subset \Delta$ defined on the set $\{|t| < \delta\} \times U \subset \mathbb{R} \times \Delta$.

Definition 1.5 (see [5, 30, 21]) *A vector field Γ_f is said to be* semi-complete *(respectively,* complete*) on Δ if the solution of the Cauchy problem (1.10) is well-defined on all of $\mathbb{R}^+ \times \Delta$ (respectively, $\mathbb{R} \times \Delta$).*

Remark 4. In fact, Theorem 1.1 asserts that *a vector field Γ_f on the (Frechét) space* $\mathrm{Hol}(\Delta, \mathbb{C})$ *is semi-complete if and only if the function f is a generator of a one-parameter continuous semigroup of holomorphic self-mappings of Δ.* In this case, the linear semigroup of composition operators $\{C_t\}_{t\geq 0} = \{C_{F_t}\}_{t\geq 0}$ is differentiable; and the semi-complete vector field Γ_f defined by (1.9) is its generator in the space $\mathrm{Hol}(\Delta, \mathbb{C})$, i.e.,

$$\lim_{t\to 0^+} \frac{1}{t}\left(I - C_t\right)(g) = \Gamma_f(g)$$

for each $g \in \mathrm{Hol}(\Delta, \mathbb{C})$, where I denotes the identity operator on $\mathrm{Hol}(\Delta, \mathbb{C})$.

Furthermore, the differential equation (1.7), rewritten as

$$\mu h = \Gamma_f h, \tag{1.11}$$

can be considered the eigenvalue problem for the linear operator Γ_f.

2 Approximation problems

2.1 A perturbation problem for spirallike function

The first approximation problem we discuss here can be described as follows. Let $f \in \mathcal{G}^+[1]$, and let $h \in \mathrm{Sp}[1]$ be a spirallike function which satisfies the equation (1.7):

$$\mu h(z) = h'(z) f(z).$$

Consider the perturbed equation

$$\mu_\tau h_\tau(z) = h'_\tau(z) f_\tau(z),$$

where $f_\tau \in \mathcal{G}[\tau]$, $\tau \in \Delta$, and $\mu_\tau = f'_\tau(\tau)$, are such that f_τ converges to f locally uniformly on Δ when τ goes to 1 unrestrictedly. (Note that such a perturbation is always possible, see, for example, Theorem 1.2.) We ask: **does the net $\{h_\tau\}$ converge to h as $\tau \to 1$?**

The following simple example shows that, in general, the answer is negative.

Example 1. Let $f(z) = (z-1) \in \mathcal{G}^+[1]$. Then the function $h(z) = 1 - z$ satisfies equation (1.7) with $\mu = 1$:

$$h(z) = h'(z) f(z).$$

Define now $f_\tau(z) = \dfrac{(z-\tau)(1-z\tau)}{1-z}$, $\tau \in (0,1)$. Obviously, $f_\tau \in \mathcal{G}[\tau]$ (by Theorem 1.2) and f_τ converges to f as $\tau \to 1^-$. Consider the perturbed problem

$$\mu_\tau h_\tau(z) = h'_\tau(z) f_\tau(z), \tag{2.1}$$

where $\mu_\tau = f'_\tau(\tau) = 1+\tau,\ f_\tau \to f$ as $\tau \to 1^-$. Then the function

$$h_\tau(z) = \frac{(\tau - z)(1 - z\tau)^{\frac{1}{\tau}}}{\tau}$$

is a solution of equation (2.1) satisfying $h_\tau(0) = h(0) = 1$. Letting τ tend to the boundary point 1, we obtain that the limit function

$$\lim_{\tau\to 1^-} h_\tau(z) = (1-z)^2,$$

which is different from $h(z)$.

At the same time if we choose $f_\tau \in \mathcal{G}[\tau]$ in a different way, say $f_\tau(z) = z - \tau$, we see that $h_\tau = 1 - \frac{z}{\tau}$ defined as a solution of equation (2.1) converges to h.

Thus one can consider the following perturbation problem. **For any $\tau \in \Delta$, find a perturbed function $f_\tau \in \mathcal{G}[\tau]$ converging to f as τ goes to 1 unrestrictedly and such that the solution of (2.1) converges to the original solution of equation (1.7) uniformly on compact subsets of Δ.**

Geometrically, an affirmative answer to this question would give us a constructive method for approximation of spirallike (starlike) functions with respect to a boundary point by spirallike (starlike) functions with respect to interior points.

We solve this problem as follows. Given τ, we find a transformation $\Phi_\tau : \mathrm{Hol}(\Delta, \mathbb{C}) \mapsto \mathrm{Hol}(\Delta, \mathbb{C})$ which takes $h \in \mathrm{Sp}[1]$ to $h_\tau \in \mathrm{Sp}[\tau],\ \tau \in \Delta$ (i.e., $h_\tau = \Phi_\tau(h)$), and such that $\Phi_\tau(h)$ tends to h when τ tends to 1 (see Theorem 3.4).

To do this, we need first to consider some approximation and interpolation problems for the classes $\mathcal{G}$ and $\mathcal{P}$ which are of independent interest.

2.2 An approximation problem for generators

As already mentioned, if a solution h of equation (1.7) is univalent, then μ must lie in the region $\Omega = \left\{ w \neq 0 :\ |w - \beta| \le \beta,\ \text{or } |w + \beta| \le \beta \right\}$, where $\beta = f'(\tau) > 0$.

Actually, the instability phenomenon we have seen in Example 1 above follows from the fact that $\mu_\tau = f'_\tau(\tau)$ does not necessarily converge to μ as

$\tau \to 1$ even if $\mu = f'(1)$. Therefore, one can pose the following problem on controlled approximation for functions in the class $\mathcal{G}[\tau],\ \tau \in \partial\Delta$.

Let $f \in \mathcal{G}^{+}[1]$ with $f'(1) = \beta > 0$. For $\tau \in \Delta$ and given $\mu \in \Omega_{+}$, find a net $\{f_\tau\},\ f_\tau \in \mathcal{G}[\tau]$, converging locally uniformly to f as $\tau \to 1$ unrestrictedly and such that $\mu_\tau = f'_\tau(\tau)$ converges to μ.

We show how to solve this problem in Section 3.

2.3 An approximation problem for semigroups

For each $f \in \mathcal{G}[\tau],\ \tau \in \overline{\Delta}$, the generated semigroup $\mathcal{S} = \{F_t\}_{t\geq 0} \in \mathcal{B}[\tau]$ converges to the point τ as $t \to \infty$. Moreover, the rate of convergence of the semigroup can be estimated by the derivative $f'(\tau)$ of the generator f at the point τ (for the boundary case $\tau \in \partial\Delta$, this fact follows by Theorem 1.3; for the general case $\tau \in \overline{\Delta}$, we refer to [13], see also [27]).

Note that the number μ in Section 2.2 is not necessarily real, while the angular derivative $f'(1)$ is a nonnegative real number. So, in light of the instability phenomena mentioned above, the following question seems to be natural.

Let $f \in \mathcal{G}^{+}[1]$ generate the semigroup $\mathcal{S} = \{F_t\}_{t\geq 0} \in \mathcal{B}[1]$. Suppose that a net $\{f_\tau\},\ f_\tau \in \mathcal{G}[\tau],\ \tau \in \Delta$, converges to f locally uniformly as $\tau \to 1$. Do the corresponding elements $F_{t,\tau},\ t \geq 0$, of the semigroups $\mathcal{S}_\tau$ generated by f_τ converge to F_t for all $t \geq 0$ as τ tends to 1?

Here the difficulty is that the value $F_{t,\tau}(z)$ must lie in a neighborhood of $\tau \in \Delta$ for any $z \in \Delta$ and for t sufficiently large, while $F_t(z)$ is close to 1.

Nevertheless, we shall show that for each $r \in (0,1)$ and $T > 0$, the net $\{F_{t,\tau}(z)\}$ converges to $F_t(z)$ uniformly on the set $[0,T] \times (r\Delta)$.

2.4 An approximation problem for functions with positive real part

By the Berkson–Porta representation (see Theorem 1.2), each cone $\mathcal{G}[\tau]$, $\tau \in \overline{\Delta}$, can be parameterized by elements of the cone $\mathcal{P}$ of functions with positive real part. It turns out that the problems in Sections 2.1–2.3 lead us to the following interpolation question about the approximation of functions of class $\mathcal{P}$. Recall that a holomorphic function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ is said to be conformal at a boundary point $\tau \in \partial\Delta$ if its angular derivative $f'(\tau)$ exists finitely and $f'(\tau) \neq 0$.

Let $q \in \mathcal{P}$ be conformal at the boundary point 1. For $\tau \in \Delta$ and given $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, does there exist a net $\{q_\tau\} \subset \mathcal{P}$ converging locally uniformly to q as $\tau \to 1$ unrestrictedly and such that

$$\arg q_\tau(\tau) = \varphi \quad \textit{for all } \tau \in \Delta.$$

A solution of this problem is given in Section 3.

3 Main results

The well-known Riesz–Herglotz formula

$$p(z) = \oint\limits_{\partial\Delta} \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\, dm_p(\zeta) + i \operatorname{Im} p(0) \tag{3.1}$$

establishes a linear one-to-one correspondence between class $\mathcal{P}$ and the set of all positive measure functions $m(= m_p)$ on the unit circle.

A consequence of this formula is that fact that for each $\tau \in \partial\Delta$ the angular limit

$$\delta_p(\tau) = \angle \lim_{z \to 1} (1 - z\overline{\tau}) p(z) = 2m_p(\tau) \tag{3.2}$$

exists and is a nonnegative real number.

Definition 3.1 *The number $\delta = \delta_p(\tau)$ defined by (3.2) is the* charge *of the function $p \in \mathcal{P}$ at the boundary point $\tau \in \partial\Delta$.*

This number plays a crucial role in our further considerations. In particular, by using the Julia–Wolff–Carathéodory Theorem, one can show that

$$\delta_p(\tau) = 2 \inf_{z \in \Delta} \frac{1 - |z|^2}{|1 - z|^2} \operatorname{Re} p(z)$$

(cf. [28]).

We denote by $\mathcal{P}^+[1]$ the subclass of $\mathcal{P}$ consisting of functions with positive charges at $\tau = 1$, i.e., $p \in \mathcal{P}^+[1]$ if and only if $\delta_p(1) > 0$. Thus, $p \in \mathcal{P}^+[1]$ if and only if the function $q \in \mathcal{P}$ defined by $q = \frac{1}{p}$ is conformal at $\tau = 1$ with $q(1) = 0$. Moreover, in this case, $q'(1)$ is a negative real number.

The following assertion, which gives a solution to the problem in Section 2.4, is the key for our further considerations.

Theorem 3.1 *Let $q \in \mathcal{P}$ be conformal at $\tau = 1$ with $q(1) = 0$. Then for all $\tau \in \Delta$ and each $\gamma \in \mathbb{C}$ such that*

$$\operatorname{Re}\gamma \geq \frac{\alpha}{2}, \quad \textit{where} \;\; \alpha = -q'(1)\,(>0)\,, \tag{3.3}$$

there exist functions $\{q_\tau\}_{\tau\in\Delta} \subset \mathcal{P}$ converging to q uniformly on compact subsets of Δ when τ tends to 1 unrestrictedly and such that

$$q_\tau(\tau) = \gamma(1 - |\tau|^2) \to 0 \quad \textit{as} \quad \tau \to 1. \tag{3.4}$$

In particular, if γ is real and $\gamma \geq \dfrac{\alpha}{2}$, the values $q_\tau(\tau)$ are real numbers.

Proof. For $\gamma \in \mathbb{C}$, consider the function

$$r(z) = \frac{zq(z) + \gamma z^2 - \bar{\gamma} - 2iz\operatorname{Im}\gamma}{-(1-z)^2}. \tag{3.5}$$

Clearly, $r \in \operatorname{Hol}(\Delta, \mathbb{C})$ and $r(0) = \bar{\gamma}$.

We claim that $r \in \mathcal{P}$ if and only if γ satisfies inequality (3.3).

Indeed, consider the function

$$f(z) = zq(z) + \gamma z^2 - \bar{\gamma} - 2iz\operatorname{Im}\gamma, \tag{3.6}$$

which is the numerator of (3.5).

It follows by Theorem 1.2 that $f \in \mathcal{G}$. In addition,

$$f(1)\left(= \angle\lim_{z\to 1} f(z)\right) = \gamma - \bar{\gamma} - 2i\operatorname{Im}\gamma = 0$$

and

$$\begin{aligned} f'(1) &\left(= \angle\lim_{z\to 1}\frac{f(z)}{z-1}\right) \\ &= q'(1) + \angle\lim_{z\to 1}\frac{\gamma z^2 - \bar{\gamma} - 2iz\operatorname{Im}\gamma}{z-1} = q'(1) + 2\operatorname{Re}\gamma. \end{aligned}$$

Thus,

$$f'(1) = 2\operatorname{Re}\gamma - \alpha \geq 0 \tag{3.7}$$

if and only if condition (3.3) holds.

At the same time, it follows from the infinitesimal version of the Julia–Wolff–Carathéodory Theorem (Theorem 1.3) that a generator $f \in \mathcal{G}$ belongs

to the subclass $\mathcal{G}[1]$ if and only if $f(1) = 0$ and $f'(1) \geq 0$. On the other hand, by the uniqueness of the Berkson–Porta representation (see Theorem 1.2) of the class $\mathcal{G}$, the inequality in (3.7) is equivalent to

$$\operatorname{Re} \frac{f(z)}{-(1-z)^2} \geq 0.$$

Comparing (3.7) with (3.5) and (3.6) proves our claim.

Now for $\tau \in \Delta$, define the function q_τ on Δ by

$$q_\tau(z) = \frac{1}{z} \left[(z - \tau)(1 - z\bar{\tau})\, r(z) + \bar{\gamma}\tau - \gamma\bar{\tau}z^2 + 2iz \operatorname{Im} \gamma \right] \tag{3.8}$$

or

$$q_\tau(z) = \frac{1}{z} g_\tau(z), \tag{3.9}$$

where

$$g_\tau(z) = (z - \tau)(1 - z\bar{\tau})\, r(z) + \bar{\gamma}\tau - \gamma\bar{\tau}z^2 + 2iz \operatorname{Im} \gamma. \tag{3.10}$$

Since $r(0) = \bar{\gamma}$, we have $g_\tau(0) = 0$. Hence, q_τ is holomorphic in Δ.

Moreover, since $\mathcal{G}$ is a real cone, we have by Theorem 1.2 (ii)–(iii) that $g_\tau \in \mathcal{G}$ for all $\tau \in \Delta$. Consequently, $g_\tau \in \mathcal{G}[0]$.

Again by Theorem 1.2 (ii), $\operatorname{Re} q_\tau(z) \left(= \operatorname{Re} \dfrac{g_\tau(z)}{z} \right) \geq 0$ for $z \in \Delta$.

In addition, since

$$q(z) = \frac{1}{z}\, g(z), \tag{3.11}$$

where

$$g(z) = -(1 - z)^2\, r(z) - \gamma z^2 + \bar{\gamma} + 2iz \operatorname{Im} \gamma, \tag{3.12}$$

we have by (3.9) and (3.10) that $q_\tau(z)$ converges to $q(z)$ as τ tends to 1 unrestrictedly.

Finally, direct calculation shows that $q_\tau(\tau) = \gamma\,(1 - |\tau|^2)$.

Example 2. Consider the function $q(z) = 1 - z$, which obviously has positive real part. Then the function r defined by (3.5) has the form

$$r(z) = \frac{1}{1 - z}.$$

Substituting this function into formula (3.8), we find the approximating functions

$$q_\tau(z) = \bar{\tau}(1 - z) + \frac{|1 - \tau|^2}{1 - z}.$$

We see that $q_\tau \to q$ as τ tends to 1 unrestrictedly.

(i) Choosing the sequence of real numbers $\tau_n^{(1)} = 1 - \frac{1}{n}$, we get the approximating sequence $q_n^{(1)}$ of functions with positive real part. In Figure 1, we see the images of the unit circle $\partial\Delta$ under the approximating functions $q_1^{(1)}$, $q_2^{(1)}$ and $q_4^{(1)}$ as well as the image of q. It is worth noting that $q_1^{(1)}(\Delta) = \{w : \operatorname{Re} w > \frac{1}{2}\}$ and that the images $q_n^{(1)}(\Delta)$ increase in the sense that $q_n^{(1)}(\Delta) \subset q_{n+1}^{(1)}(\Delta)$. As n tends to infinity, these images tend to the right half-plane, while the original function q is bounded. (Note that the Carathéodory Kernel Theorem is not applicable here since the functions q_n are not univalent.)

(ii) Choosing another sequence $\tau_n^{(2)}$ converging to 1, say, $\tau_n^{(2)} = 1 - \frac{3(1-i)}{n}$, we get a different approximating sequence $q_n^{(2)}$ of positive real part functions. In Figure 2, we see the images of the unit circle $\partial\Delta$ under the functions $q_4^{(2)}$, $q_6^{(2)}$ and $q_{12}^{(2)}$. Once again, in spite of the boundedness of q, for n large enough the image $q_n^{(2)}(\Delta)$ almost covers the right half-plane.

Corollary 3.1 *Let $p \in \mathcal{P}^+[1]$ with the charge $\delta_p(1) = \beta > 0$. Then for all $\tau \in \Delta$ and each $\mu \in \mathbb{C}$ such that*

$$|\mu - \beta| \leq \beta, \;\; \mu \neq 0, \tag{3.13}$$

there exist functions $\{p_\tau\}_{\tau\in\Delta} \subset \mathcal{P}$ which converge to p uniformly on compact subsets of Δ as τ tends to 1 and such that

$$p_\tau(\tau) = \frac{\mu}{1 - |\tau|^2} \to \infty. \tag{3.14}$$

In particular, if μ is real and

$$0 < \mu \leq 2\beta, \tag{3.15}$$

the values $p_\tau(\tau)$ are real numbers.

Proof. Setting

$$q(z) = \frac{1}{p(z)}\,,$$

we have $q \in \mathcal{P}$ with $q(1) = 0$ and

$$q'(1) = \angle\lim_{z\to 1} \frac{q(z)}{z-1} = \angle\lim_{z\to 1} \frac{1}{(z-1)p(z)} = -\frac{1}{\beta}\,(:= -\alpha)\,.$$

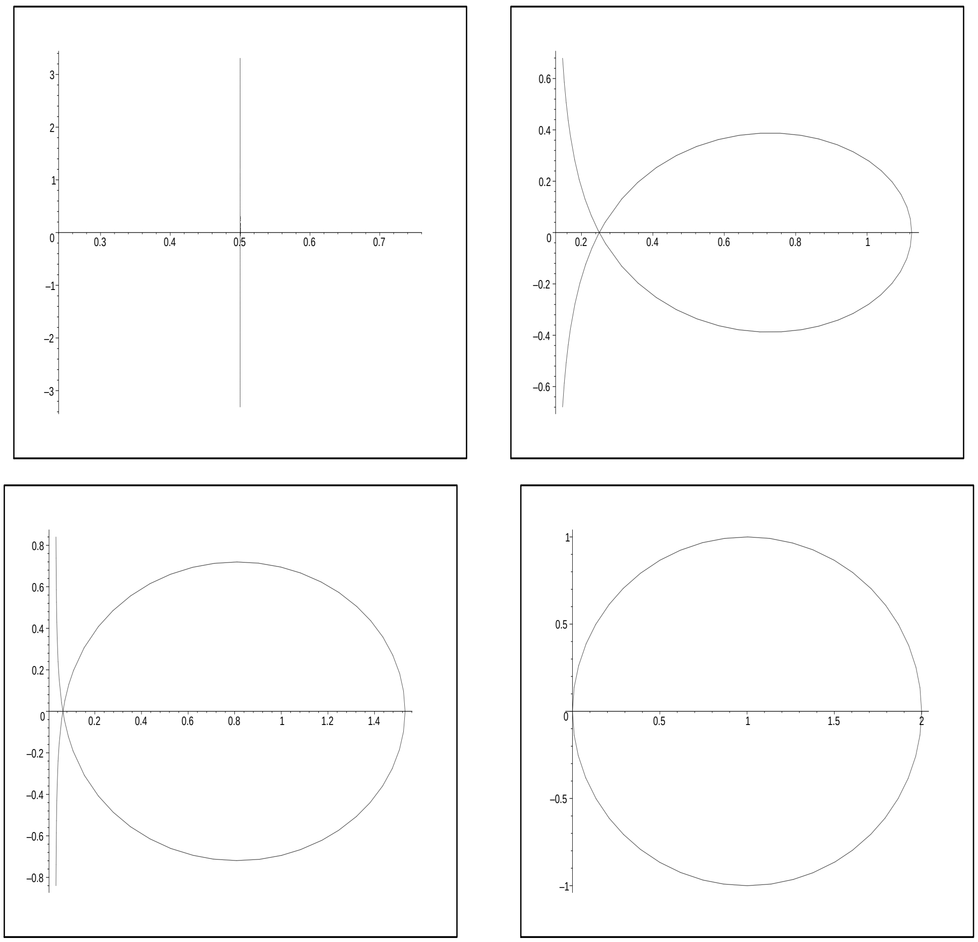

Figure 1: Example 2 (i), the images of $q_1^{(1)},\ q_2^{(1)},\ q_4^{(1)}$ and of q.

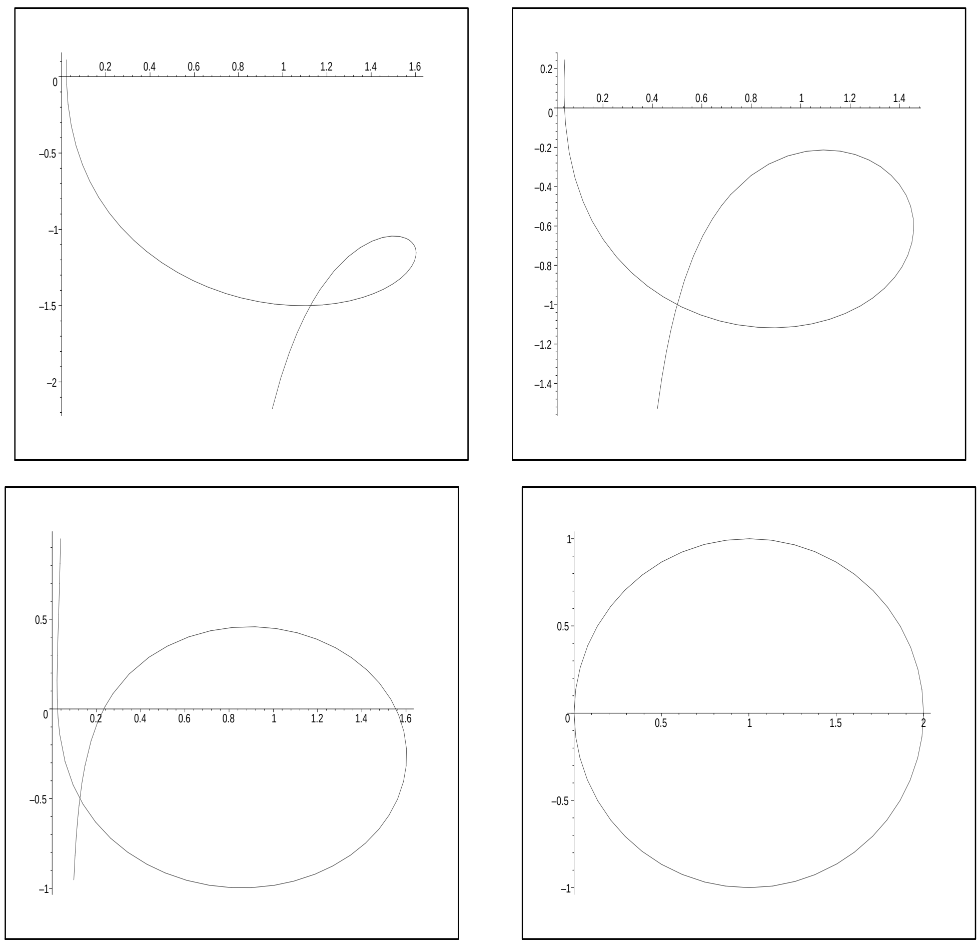

Figure 2: Example 2 (ii), the images of $q_4^{(2)},\ q_6^{(2)},\ q_{12}^{(2)}$.

Using Theorem 3.1 for each $\gamma \in \mathbb{C}$ such that

$$2\,\mathrm{Re}\,\gamma \geq \frac{1}{\beta} \tag{3.16}$$

and all $\tau \in \Delta$, one can find $q_\tau \in \mathcal{P}$ converging to q as $\tau \to 1$ and such that

$$q_\tau(\tau) = \gamma\left(1 - |\tau|^2\right).$$

Setting now

$$p_\tau(z) = \frac{1}{q_\tau(z)} \tag{3.17}$$

and $\gamma = \frac{1}{\mu}$, we obtain that (3.13) is equivalent to (3.16) and p_τ converges locally uniformly to p.

Finally,

$$p_\tau(\tau) = \frac{1}{q_\tau(\tau)} = \frac{1}{\gamma\,(1 - |\tau|^2)} = \frac{\mu}{1 - |\tau|^2}\,,$$

and we are done.

It is now easy to prove the following theorem, which solves the problem in Section 2.2.

Theorem 3.2 *Let $f \in \mathcal{G}$ satisfy the conditions $f(1) = 0$ and $f'(1) = \beta > 0$, i.e., $f \in \mathcal{G}^+[1]$. Then for each $\mu \in \mathbb{C}$ such that*

$$|\mu - \beta| \leq \beta,\ \ \mu \neq 0 \tag{3.18}$$

and for all $\tau \in \Delta$, there exist generators $\{f_\tau\}$ with $f_\tau \in \mathcal{G}[\tau]$ such that $f'_\tau(\tau) = \mu$ and f_τ converges to f uniformly on compact subsets of Δ as τ tends to 1 unrestrictedly.

In particular, setting $\mu = \beta$ we have $f'_\tau(\tau) = \beta \in (0, \infty)$ for all $\tau \in \Delta$.

Proof. By the Berkson–Porta formula (see Theorem 1.2), f must be of the form

$$f(z) = -\left(1 - z\right)^2 p(z), \tag{3.19}$$

where $p \in \mathcal{P}^+[1]$ with $\delta_p(1) = \beta > 0$.

By Corollary 3.1, for each μ satisfying (3.13) and $\tau \in \Delta$, one can find $p_\tau \in \mathcal{P}$ with $p_\tau(\tau) = \dfrac{\mu}{1 - |\tau|^2}$ and such that p_τ converges to p as $\tau \to 1$.

Now we construct $f_\tau \in \mathcal{G}$ by the Berkson–Porta representation (1.5)

$$f_\tau(z) = (z-\tau)\,(1-z\bar{\tau})\,p_\tau(z).$$

Since $f'_\tau(\tau) = (1-|\tau|^2)\,p_\tau(\tau)$, we obtain our assertion.

In some sense a converse assertion is also true.

Theorem 3.3 *Let $f \in \mathcal{G}^+[1]$, i.e., $f(1) = 0$ and $f'(1) = \beta > 0$, and let $f_n \in \mathcal{G}$ vanish at $\tau_n \in \Delta$. Suppose that f_n converges to f uniformly on compact subsets of Δ. Then*

(i) the sequence $\{\tau_n\}$ converges to 1;

(ii) if the sequence $\mu_n := f'_n(\tau_n)$ converges to $\mu \neq 0$, then μ must satisfy condition (3.13), i.e., $|\mu - \beta| \leq \beta$.

Proof. (i) For each $n = 1, 2, \ldots$, the function $f_n \in \mathcal{G}[\tau_n]$ has the form $f_n(z) = (z-\tau_n)\,(1-z\overline{\tau_n})\,p_n(z)$ with $\operatorname{Re} p_n(z) \geq 0$ everywhere.

Since $\overline{\Delta}$ is a compact subset of $\mathbb{C}$ and $\{p_n\}$ is a normal family on Δ, one can choose a subsequence $\{n_k\} \subset \mathbb{N}$ such that τ_{n_k} converges to $\tau \in \overline{\Delta}$ and p_{n_k} converges either to $\widetilde{p} \in \mathcal{P}$ or $\widetilde{p} = \infty$. In any case, the sequence $\{f_{n_k}\} \subset \mathcal{G}$ converges to either $f(z) = (z-\tau)\,(1-z\bar{\tau})\,\widetilde{p}(z)$ with $\operatorname{Re}\widetilde{p}(z) \geq 0$ and $\tau \in \overline{\Delta}$ or to infinity. Since the latter case is impossible, the uniqueness of the Berkson–Porta representation and (3.19) imply that $\tau = 1$ and $\widetilde{p} = p$.

(ii) Assume now that $\mu_n := f'_n(\tau_n)$ converges to $\mu \in \mathbb{C}$, $\mu \neq 0$, and consider the differential equations

$$\mu_n h_n(z) = h'_n(z) f_n(z), \tag{3.20}$$

normalized by the conditions $h_n(0) = 1$. This initial value problem has a unique solution which is a univalent function on Δ spirallike with respect to an interior point and $h_n(\tau_n) = 0 \in h_n(\Delta)$.

Explicitly, h_n can be written as

$$h_n(z) = \exp\left[\mu_n \int_0^z \frac{dz}{f_n(z)}\right]. \tag{3.21}$$

Now for each $r \in (0,1)$, one can find $n_0 \in \mathbb{N}$ such that for all $n > n_0$, the points τ_n do not belong to the closed disk $\overline{\Delta_r} = \{|z| \leq r < 1\}$. In other words, for all $n > n_0$ the functions f_n do not vanish in this disk. Then the functions h_n defined by (3.21) converge to a function $h \in \operatorname{Hol}(\Delta_r, \mathbb{C})$ uniformly on this

disk. Since r is arbitrary, we have that actually $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ and has the form

$$h(z) = \exp\left[\mu \int_0^z \frac{dz}{f(z)}\right]. \tag{3.22}$$

In addition, it follows by Hurwitz's theorem that h is either a univalent function on Δ or a constant. The latter case is impossible because of the equality $\dfrac{h'(z)}{h(z)} = \dfrac{\mu}{f(z)}$, which follows by (3.22) and $\mu \neq 0$.

On the other hand, we already know that the initial value problem

$$\beta \widetilde{h}(z) = \widetilde{h}'(z) f(z), \quad h(0) = 1$$

has also the unique solution

$$\widetilde{h}(z) = \exp\left[\beta \int_0^z \frac{dz}{f(z)}\right], \tag{3.23}$$

which is a starlike function with respect to a boundary point $\left(\widetilde{h}(1) = 0\right)$.

Comparing (3.22) and (3.23), we obtain

$$h(z) = \left[\widetilde{h}(z)\right]^{\frac{\mu}{\beta}}. \tag{3.24}$$

In addition, the Visser–Ostrowski condition

$$\angle \lim_{z \to 1} \frac{(z-1)\widetilde{h}'(z)}{\widetilde{h}(z)} = 1$$

(see [12]) implies that the smallest wedge which contains $h(\Delta)$ is exactly of angle π. Since h is univalent, we get that $\mathrm{Re}\,\dfrac{\beta}{\mu} \geq \dfrac{1}{2}$, which is equivalent to (3.13).

As a consequence of Theorems 3.2 and 3.3, we obtain the following result.

Theorem 3.4 *Let $f \in \mathcal{G}^+[1]$ with $f'(1) = \beta$. Then the initial value problem*

$$\lambda h(z) = h'(z) f(z), \quad h(0) = 1 \tag{3.25}$$

has a univalent solution $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ if and only if the complex number λ belongs to the set $\Omega = \Omega_+ \cup \Omega_-$, where

$$\Omega_\pm = \{\omega \in \mathbb{C} : \ |\omega \mp \beta| \leq \beta, \ \omega \neq 0\}. \tag{3.26}$$

Moreover,

• *for each $\mu \in \Omega_+$ and each $\tau \in \Delta$, there are $h_\tau \in \mathrm{Sp}[\tau]$ which converge to the function*

$$h_1(z) = h^{\frac{\mu}{\lambda}}(z) \tag{3.27}$$

as τ tends to 1 unrestrictedly. In particular, if $\lambda \in \Omega_+$, then choosing $\mu = \lambda$, we find $h_\tau \in \mathrm{Sp}[\tau]$ which converge to the original function h as $\tau \to 1$.

• *for each $\mu \in \Omega_-$ and for each $\tau \in \Delta$, there are meromorphic functions $\widetilde{h}_\tau$ with a unique simple pole at τ and such that $\widetilde{h}_\tau$ converges to the holomorphic function*

$$\widetilde{h}(z) = h^{\frac{\mu}{\lambda}}(z) \tag{3.28}$$

as τ tends to 1 unrestrictedly.

Proof. Take any $\mu \in \Omega_+$ and any $\tau \in \Delta$. By Theorem 3.2, one can choose generators $f_\tau \in \mathcal{G}[\tau]$ such that the functions f_τ converge uniformly on compact subsets to f and satisfy the conditions $f_\tau(\tau) = 0$ and $f'_\tau(\tau) = \mu$. Then, as in the proof of the second part of Theorem 3.3, one shows that functions $h_\tau \in \mathrm{Sp}[\tau]$ defined by

$$h_\tau = \exp\left[\mu \int_0^z \frac{dz}{f_\tau(z)}\right] \tag{3.29}$$

converge to a univalent function

$$h_1(z) = \exp\left[\mu \int_0^z \frac{dz}{f(z)}\right] \tag{3.30}$$

which satisfies the equation

$$\mu h_1(z) = h'_1(z) f(z). \tag{3.31}$$

If now $\lambda \in \Omega_+$, then setting $\mu = \lambda$, we see that $h = h_1$ must be univalent. If $\lambda \in \Omega_-$, then setting $\mu = -\lambda$, and comparing differential equations (3.25) and (3.31), we see that $h = h_1^{-1}$. Since $h_1(z) \neq 0,\ z \in \Delta$, h is a well-defined univalent function on Δ. In addition, it is clear that h_1^{-1} is a locally uniform limit of meromorphic functions h_τ^{-1} with poles at τ.

Conversely, assume that for some $\lambda \in \mathbb{C},\ \lambda \neq 0$, equation (3.25) has a univalent solution in Δ. If $\mathrm{Re}\,\lambda > 0$, then setting $\mu = \beta$ in (3.30), we see as in the proof of Theorem 3.3 that the image of the function $h_1(z) = h^{\frac{\beta}{\lambda}}(z)$

must lie in the wedge of the angle π, which is the smallest one containing $h_1(\Delta)$. Then, by a result of [3], we have $\lambda \in \Omega_+$. If $\operatorname{Re}\lambda < 0$, then the same considerations show that $-\lambda \in \Omega_+$, and we are done.

Theorem 3.5 (Perturbation formula) *Let $f \in \mathcal{G}^+[1]$ with $f'(1) = \beta$ and let $\lambda \in \Omega_+ = \{w \in \mathbb{C} : \ |w - \beta| \le \beta,\ w \neq 0\}$.*

Assume that $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ is the solution of equation (3.25)

$$\lambda h(z) = h'(z) f(z)$$

normalized by the conditions $h(0) = 1$ and $h(1) = 0$. Then for each $\mu \in \Omega_+$ and for each $\tau \in \Delta$, the function $h_\tau \in \mathrm{Hol}(\Delta, \mathbb{C})$ defined by

$$h_\tau(z) = [h(z)]^{\frac{\mu}{\lambda}}\ \frac{(z-\tau)(1-z\bar{\tau})^{\mu/\bar{\mu}}}{-(1-z)^{1+\mu/\bar{\mu}}}$$

is univalent on Δ and belongs to the class $\mathrm{Sp}[\tau]$ with $h_\tau(\tau) = 0$ and $h_\tau(0) = \tau$.

If, in particular, $\mu = \lambda$, then h_τ converges to h whenever τ tends to 1. Thus h is a univalent function on Δ spirallike (starlike) with respect to a boundary point with $h(1) = 0$.

Proof. Let $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ be a solution of the differential equation (3.25) with $f \in \mathcal{G}^+[1]$ defined by $f(z) = \dfrac{-(1-z)^2}{q(z)}$, where

$$\operatorname{Re} q(z) > 0 \quad \text{and} \quad \angle \lim_{z\to 1} \frac{1-z}{q(z)} = \beta.$$

Then

$$\frac{h'(z)}{\lambda h(z)} = \frac{q(z)}{-(1-z)^2}\ . \tag{3.32}$$

As in the proof of Theorem 3.1, for γ complex with $2\operatorname{Re}\gamma \ge \frac{1}{\beta}$ we define $r \in \mathcal{P}^+[1]$ by

$$r(z) = \frac{zq(z) + (\gamma z + \bar{\gamma})(z-1)}{-(1-z)^2}\,. \tag{3.33}$$

Comparing (3.32) and (3.33), we have

$$r(z) = \frac{zh'(z)}{\lambda h(z)} + \frac{\gamma\, z + \bar{\gamma}}{1-z}\,. \tag{3.34}$$

Once again, for each point $\tau \in \Delta$, we define

$$q_\tau(z) = \frac{1}{z}\left[(z-\tau)(1-z\bar{\tau})r(z) + \bar{\gamma}\tau - \gamma\bar{\tau}z^2 + (\gamma-\bar{\gamma})z\right].$$

Then the differential equation

$$\mu h_\tau(z) = h_\tau'(z)(z-\tau)(1-z\bar{\tau})\frac{1}{q_\tau(z)} \tag{3.35}$$

has a holomorphic solution h_τ if and only if $\mu = \dfrac{1}{\gamma} \in \Omega_+$. Moreover, $h_\tau \in \mathrm{Sp}[\tau]$ by Theorems 1.4 and 1.2. Since $h_\tau(\tau) = 0$, the function $g_\tau(z) := \dfrac{h_\tau(z)}{z-\tau}$ is well-defined. Now, using (3.34) and (3.35), we calculate

$$\begin{aligned}
\frac{g_\tau'(z)}{g_\tau(z)} &= \mu\left[\frac{h'(z)}{\lambda h(z)} + \frac{\gamma z + \bar{\gamma}}{z(1-z)} - \frac{\bar{\gamma}}{z(1-z\bar{\tau})}\right] \\
&= \mu\left[\frac{h'(z)}{\lambda h(z)} + \frac{\gamma}{1-z} + \frac{\bar{\gamma}}{1-z} + \frac{\bar{\gamma}(-\bar{\tau})}{1-z\bar{\tau}}\right] \\
&= \frac{\mu h'(z)}{\lambda h(z)} + \frac{1}{1-z} + \frac{\mu\bar{\gamma}}{1-z} + \frac{\mu\bar{\gamma}(-\bar{\tau})}{1-z\bar{\tau}}.
\end{aligned}$$

Then

$$h_\tau(z) = (z-\tau)g_\tau(z) = C\,(h(z))^{\frac{\mu}{\lambda}} \cdot \frac{(z-\tau)(1-z\bar{\tau})^{\frac{\mu}{\bar{\mu}}}}{(1-z)^{1+\frac{\mu}{\bar{\mu}}}}$$

with some constant C. So, to satisfy the normalization $h_\tau(0) = \tau$, we must set $C = -1$. The proof is complete.

Example 2. Consider the starlike function with respect to a boundary point

$$h(z) = (1-z)^{0.8}.$$

It satisfies equation (3.25) with $f(z) = z - 1$ and $\lambda = 0.8$:

$$0.8h(z) = h'(z)\cdot(z-1).$$

Setting $\mu = \lambda = 0.8$, we see that by Theorem 3.5, h can be approximated by the functions

$$h_\tau(z) = \frac{(\tau-z)(1-z\bar{\tau})}{(1-z)^{1.2}}.$$

(i) Choosing, in particular, the sequence of real numbers $\tau_n^{(1)} = 1 - \frac{3}{n}$, we get the approximating sequence $h_n^{(1)}$ of starlike functions with respect to (different) interior points. In Figure 3, we see the images of h as well as the images of the approximating functions $h_6^{(1)}$, $h_{10}^{(1)}$ and $h_{30}^{(1)}$. Note that in this case, the intersection $\bigcap_{n=0}^{\infty} h_n^{(1)}(\Delta)$ contains the left half-plane, while the image $h(\Delta)$ lies in the right half-plane.

(ii) Choosing another sequence $\tau_n^{(2)}$ converging to 1, say, $\tau_n^{(2)} = 1 - (1-i)\frac{3}{n}$, we get a different approximating sequence $h_n^{(2)}$ of starlike functions with respect to interior points. In Figure 4, we see the images of h and of the approximating functions $h_6^{(2)}$, $h_{10}^{(2)}$ and $h_{30}^{(2)}$. Once again, all of the images $h_n^{(2)}(\Delta)$ contain the left half-plane.

(iii) On the other hand, setting $\mu = 1 + i$ and choosing $\tau_n^{(3)} = 1 - \frac{3}{n}$, we find the sequence $h_n^{(3)}$ of spirallike functions with respect to interior points, which converges to the function

$$\widetilde{h}(z) = (h(z))^{\frac{1+i}{0.8}} = (1-z)^{1+i},$$

which is spirallike with respect to a boundary point. In Figure 5, one can see images of a number of approximating functions and the image of $\widetilde{h}$.

Since $\mathcal{G}^+[1]$ is a real cone, we can always consider equation (3.25) with f normalized by $f'(1) = 1$.

Definition 3.2 *We say that $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ belongs to the class $\mathrm{Sp}_\lambda[1]$ if it satisfies the equation*

$$\lambda h(z) = h'(z) f(z),$$

where $f \in \mathcal{G}^+[1]$ with $f'(1) = 1$ and $\lambda \in \Omega_+ = \{w : |w - 1| \leq 1,\ w \neq 0\}$.

It follows by Theorem 3.4 that

$$\mathrm{Sp}[1] = \bigcup_{\lambda \in \Omega_+} \mathrm{Sp}_\lambda[1]$$

and

$$S^*[1] = \bigcup_{0 < \lambda \leq 2} \mathrm{Sp}_\lambda[1].$$

In fact, for $\lambda \in (0, 2]$, the class $\mathrm{Sp}_\lambda[1]$ consists of starlike functions h such that the smallest wedge which contains the image $h(\Delta)$ is exactly of angle $\lambda\pi$.

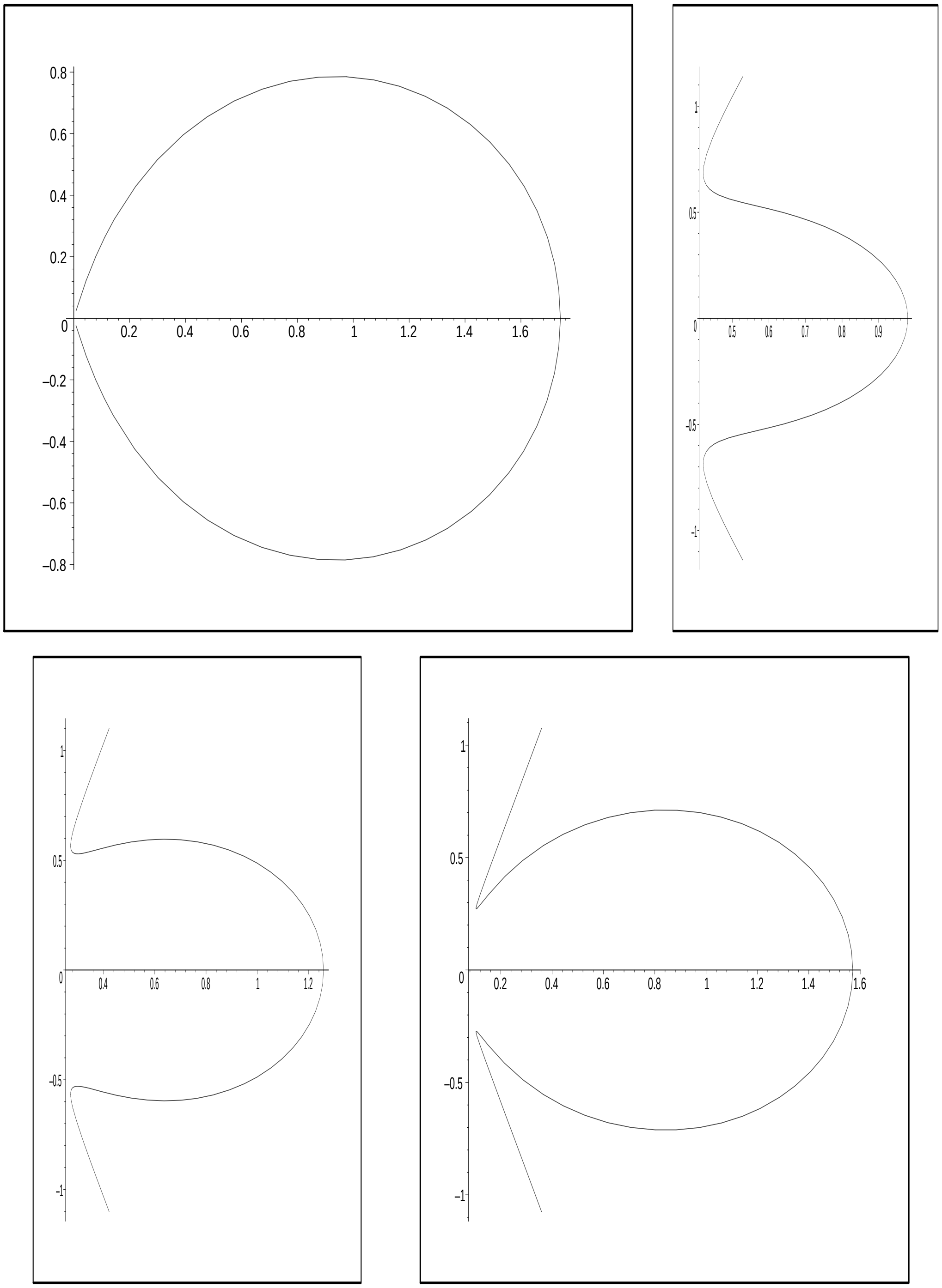

Figure 3: Example 3 (i), the images of $h, h_6^{(1)}, h_{10}^{(1)}$ and $h_{30}^{(1)}$.

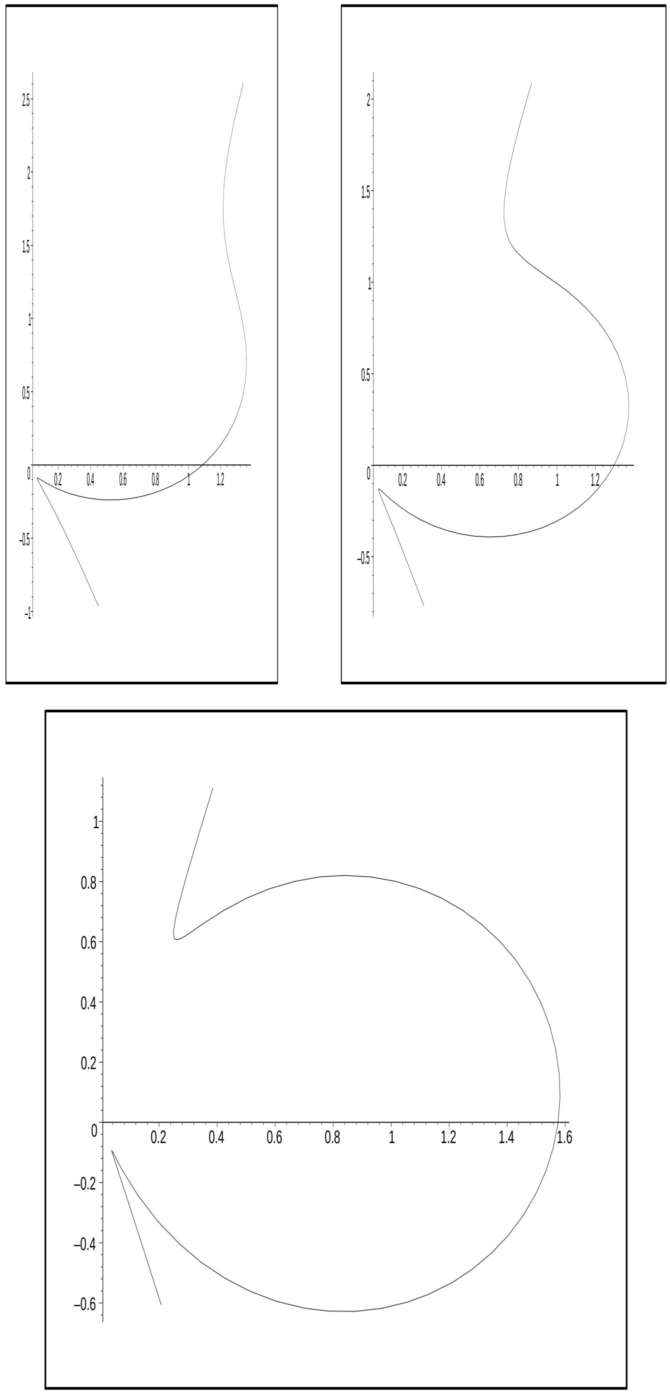

Figure 4: Example 3 (ii), the images of $h_6^{(2)}, h_{10}^{(2)}$ and $h_{30}^{(2)}$.

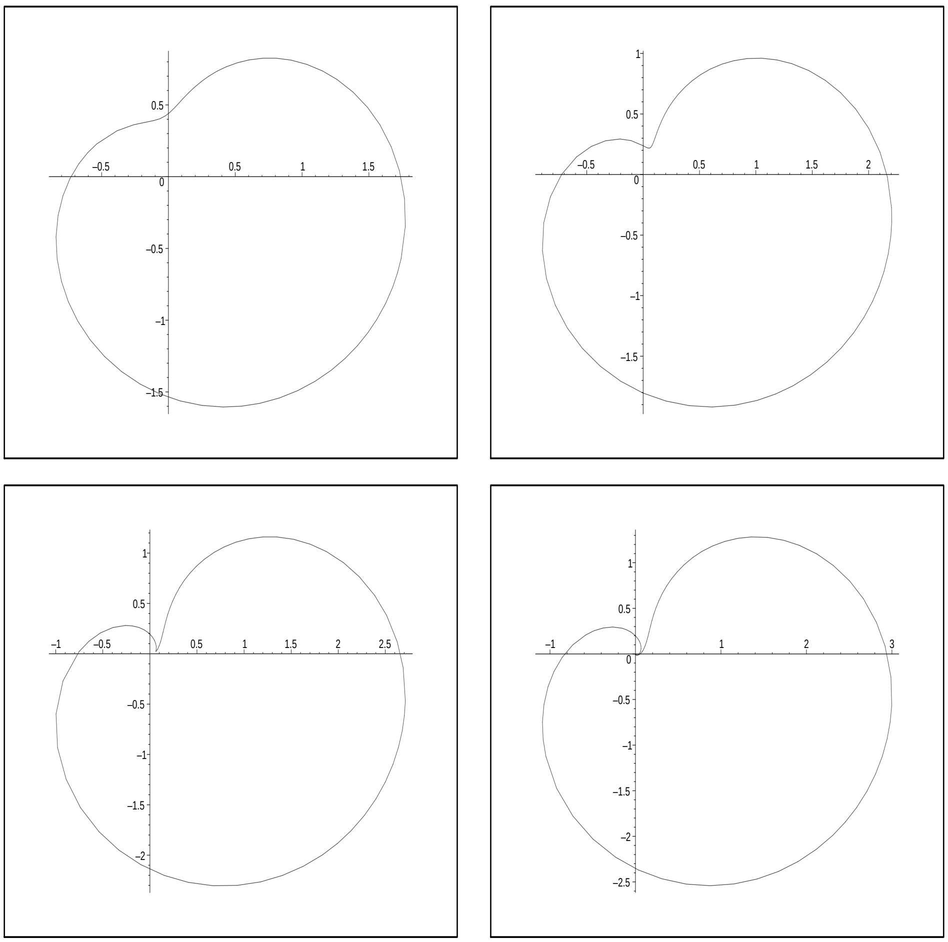

Figure 5: Example 3 (iii), the images of $h_6^{(3)}, h_{10}^{(3)}, h_{30}^{(3)}$ and $\widetilde{h}$.

Theorem 3.6 *Let $\lambda \in \Omega_+ = \{w : \ |w-1| \le 1\}$, $\lambda \ne 0$. Then a function $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ belongs to the class $\mathrm{Sp}_\lambda[1]$ if and only if it admits the representation*

$$h(z) = (1-z)^{2\lambda} \left[\frac{h_*(z)}{z}\right]^\lambda, \tag{3.36}$$

where h_0 is a starlike function of class $S^*[0]$ which satisfies the condition

$$2 \inf \left(\frac{1-|z|^2}{|1-z|^2} \operatorname{Re} \frac{z h_*'(z)}{h_*(z)} \right) = 1.$$

Proof. Let $h \in \mathrm{Sp}[1]$ be λ-spirallike. Then it satisfies the equation

$$\lambda h'(z) = h(z) f(z) \left(= -h(z)(1-z)^2 \cdot \frac{1}{q(z)} \right),$$

where $f(z) \in \mathcal{G}^+[1]$ with $f'(1) = 1$ (or, what is the same, $q'(1) = -1$ and $q \in \mathcal{P}$).

By the above theorem, the functions $h_\tau \in \mathrm{Hol}(\Delta, \mathbb{C})$ defined by

$$h_\tau(z) = h^{\frac{\mu}{\lambda}}(z) \, \frac{(z-\tau)(1-z\bar{\tau})^{\frac{\mu}{\bar{\mu}}}}{-(1-z)^{1+\frac{\mu}{\bar{\mu}}}}$$

are spirallike with respect to an interior point ($h(\tau) = 0$) and satisfy the equations

$$\mu h_\tau(z) = h_\tau'(z)(z-\tau)(1-z\bar{\tau}) \, \frac{1}{q_\tau(z)} \, ,$$

where $q_\tau \in \mathcal{P}$ and defined by the formula

$$q_\tau(z) = \frac{1}{z} \, \left[(z-\tau)(1-z\bar{\tau}) r(z) + \bar{\gamma}\tau - \gamma\bar{\tau} z^2 + 2iz \operatorname{Im} \gamma \right],$$

where

$$r(z) = \frac{z q(z) + \gamma z^2 - \bar{\gamma} - 2iz \operatorname{Im}\gamma}{-(1-z)^2} \, , \ \gamma = \frac{1}{\mu} \, .$$

If, in particular, we set $\mu = 1$ and $\tau = 0$, then we get $\gamma = 1$; and $h_0(z) = h^{\frac{1}{\lambda}} \cdot \frac{z}{-(1-z)^2}$, hence $h_* = -h_0$, is of the class $S^*[0]$ and satisfies the equation

$$h_*(z) = h_*'(z) f_0(z) = h_*'(z) \cdot \frac{z}{q_0(z)},$$

where $q_0(z) = r(z)$.

Since

$$\delta_r(1) = \angle \lim_{z\to 1}(1-z)r(z) = \angle \lim_{z\to 1} \frac{zq(z)}{z-1} + \angle \lim_{z\to 1} \frac{z^2-1}{z-1} = 1,$$

we obtain

$$\operatorname{Re} q_0(z) = \operatorname{Re} \frac{zh_*'(z)}{h_*(z)} \geq \frac{1}{2} \frac{1-|z|^2}{|1-z|^2}.$$

Conversely, let $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ admits the representation

$$h(z) = (1-z)^{2\lambda} \left[\frac{h_*(z)}{z}\right]^{\lambda},$$

where h_0 satisfies the required condition. Differentiating logarithmically, we obtain

$$\frac{1}{\gamma} \frac{h'(z)}{h(z)} = -\frac{2}{1-z} + \frac{1}{z}\left[\frac{zh_0'(z)}{h_0(z)} - 1\right] = -\frac{2}{1-z} + \frac{1}{z}\left[q_0(z) - 1\right].$$

Consider the function $q \in \mathrm{Hol}(\Delta, \mathbb{C})$ defined by

$$q(z) = -\frac{(1-z)^2}{\lambda} \frac{h'(z)}{h(z)}.$$

We have to show that $q \in \mathcal{P}$ and $q'(1) = -1$. This will imply that h satisfies Definition 3.2 with $f(z) = -(1-z)^2 \frac{1}{q(z)}$.

Indeed, if $g(z) = zq(z)$, we have

$$g(z) = 1 - z^2 - (1-z)^2 q_0(z).$$

Both terms in the right-hand side of this equality are in $\mathcal{G}$. Hence g also belongs to $\mathcal{G}$, since $\mathcal{G}$ is a real cone.

Thus q must belong to $\mathcal{P}$ by the Berkson–Porta formula (1.5) with $\tau = 0$.

Finally, a direct calculation shows that

$$q'(1) = \angle \lim_{z\to 1} \frac{q(z)}{z-1} = \angle \lim_{z\to 1} \frac{1-z^2}{z-1} + \angle \lim_{z\to 1}(1-z)q_0(z) = -1,$$

and we are done.

Define $\widetilde{S}^*[0]$ as the subclass of $S^*(= S^*[0])$ of all starlike functions $h \in \mathrm{Hol}(\Delta, \mathbb{C})$, $h(0) = 0$, which satisfy the condition

$$2 \inf \frac{1-|z|^2}{|1-z|^2} \operatorname{Re} \frac{zh'(z)}{h(z)} = 1.$$

Corollary 3.2 *For each* $\lambda \in \Omega_+ = \{w : |w-1| \le 1,\ w \ne 0\}$*, the class* $\widetilde{S}^*[0]$ *is homeomorphic to the class* $\mathrm{Sp}_\lambda[1]$.

Using representation (3.36), one checks easily the following characterization of spirallike functions with respect to a boundary point (see [24, 19, 29]).

Corollary 3.3 (Generalized Robertson condition) *A function* $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ *normalized by the condition* $h(0) = 1$ *is of class* $\mathrm{Sp}[1]$ *if and only if for some* $\lambda \in \Omega_+$

$$\mathrm{Re}\left[\frac{2z}{\lambda}\frac{h'(z)}{h(z)} + \frac{1+z}{1-z}\right] \ge 0.$$

Using Theorem 3.4 above, one can prove the uniform stability of a semigroup generated by $f \in \mathcal{G}^+[1]$ under any perturbation of its generator (see Problem in Section 2.3).

Let $\{F_t\}_{t\ge0}$ be a semigroup generated by $f \in \mathcal{G}^+[1]$ and let $f_n \in \mathcal{G}[\tau_n]$, $\tau_n \in \Delta$, be a sequence of generators converging to f uniformly on compact subsets of Δ. If $\left\{F_t^{(n)}\right\}_{t\ge0}$ is a semigroup generated by f_n, then it follows that $F_t^{(n)}(z) \to \tau_n$ for each z as t tends to ∞. In addition, since f_n and f are locally Lipshitzian on Δ, the uniqueness property of the Cauchy problem implies that for each $a \in \Delta$, there exist $r > 0$ and $T(= T(r))$ such that for each z from the circle $|z-a| < r$ and each $t \in [0, T(r)]$, the sequence of values $F_t^{(n)}(z)$ converges to $F_t(z)$ as n tends to infinity. The question is whether this convergence is uniform on each compact subset of $\Delta \times \mathbb{R}^+$, in other words, whether for each $0 < r < 1$ and each $0 < T < \infty$ which does not depend on r, the sequence $F_t^{(n)}$ converges to F_t uniformly on the set $\overline{\Delta_r} \times [0, T]$, where $\overline{\Delta_r} = \{z \in \mathbb{C} : \ |z| \le r\}$.

We now show that the answer is affirmative.

Theorem 3.7 *Let* $\{F_t\}_{t\ge0}$ *be a semigroup generated by* $f \in \mathcal{G}^+[1]$*, and let* $f_n \subset \mathcal{G}$ *be a sequence of generators converging to* f *uniformly on compact subsets of* Δ *such that* $f_n(\tau_n) = 0$ *with* $\tau_n \in \Delta$*. Assume also that the closure of the set* $\{\mu_n = f_n'(\tau_n)\}$ *does not contain the origin. Then the sequence* $\left\{F_t^{(n)}(z)\right\}_{t\ge0}$ *of semigroups generated by* f_n *converges to* $F_t(z)$ *uniformly on compact subsets of* $\Delta \times \mathbb{R}^+$.

Proof. By passing to a subsequence, if necessary, we may suppose that the sequence $\mu_n = f_n'(\tau_n)$ is convergent, say, to $\mu \in \mathbb{C}$. It follows from our assumption that $\mu \neq 0$ and Theorem 3.3 that μ must satisfy the condition $|\mu - \beta| \leq \beta$. Then the sequence $\{h_n\}$ defined by

$$h_n(z) = \exp\left[\mu \int_0^z \frac{dz}{f_\tau(z)}\right]$$

converges to the function h_1 defined by (3.30).

In addition, we have

$$F_t^{(n)}(z) = h_n^{-1}\left(e^{-t\mu_n} h_n(z)\right). \tag{3.37}$$

Thus $F_t^{(n)}(z)$ converges uniformly on compact subsets of $\Delta \times \mathbb{R}^+$ to a function $G: \Delta \times \mathbb{R}^+ \mapsto \Delta$ defined by

$$G(t, z) = h_1^{-1}\left(e^{-t\mu} h_1(z)\right). \tag{3.38}$$

On the other hand, we already know that if h is a solution of the equation $\beta h(z) = h'(z) f(z)$, then

$$F_t(z) = h^{-1}\left(e^{-t\beta} h(z)\right) \tag{3.39}$$

and

$$h_1 = h^{\frac{\mu}{\beta}}. \tag{3.40}$$

Then we get from (3.38) and (3.40)

$$h^{\frac{\mu}{\beta}}\left(G(t, z)\right) = e^{-t\mu} h^{\frac{\mu}{\beta}}(z)$$

or

$$h\left(G(t, z)\right) = e^{-t\beta} h(z).$$

The latter equation and (3.39) imply that $G(t, z) = F_t(z)$ for all $z \in \Delta$ and $t \in \mathbb{R}^+$.

Another useful application of the results given above is the following description of the spectrum of the infinitesimal generator of a one-parameter semigroup of composition operators in the infinite dimensional Frechét space $E = \mathrm{Hol}(\Delta, \mathbb{C})$.

For $f \in \mathcal{G}$, define a linear operator Γ_f on E by the formula

$$\left(\Gamma_f h\right)(z) := h'(z) f(z) \tag{3.41}$$

(see Section 1.3).

Recall that the spectrum $\sigma(\Gamma_f)$ of the operator Γ_f is the set of all complex numbers $\lambda \in \mathbb{C}$ for which $\lambda I - \Gamma_f$ is not continuously invertible, where I is the identity operator on $\mathrm{Hol}(\Delta, \mathbb{C})$. The point spectrum $\sigma_p(\Gamma_f)$ of Γ_f is the subset of $\sigma(\Gamma_f)$ which consists of its eigenvalues, i.e.,

$$\sigma_p(\Gamma_f) = \{\lambda \in \mathbb{C} : \ (\lambda I - \Gamma_f)\, h = 0 \text{ for some } h \neq 0\}.$$

In this case, a nontrivial solution of the equation $(\lambda I - \Gamma_f)\, h = 0$ is called an eigenvector of Γ_f corresponding to eigenvalue λ.

Generally speaking, in infinite dimensional spaces, the spectrum of a linear operator does not coincide with its point spectrum. However, since in our situation $f \in \mathcal{G}^+[1]$ does not vanish in Δ, we have $\sigma_p(\Gamma_f) = \sigma(\Gamma_f) = \mathbb{C}$.

This, for example, implies immediately that the linear semigroup generated by Γ_f (and hence, the nonlinear semigroup $\mathcal{S} = \{F_t\}_{t \geq 0}$ generated by f) cannot be holomorphically extended into a sector containing the positive real axis (see, for example, [31, Chapter IX.10]).

Over the past few decades a lot of work address a wide range of topics concerning properties of composition operators on classical Banach spaces of analytic functions. In particular, composition operators have been studied extensively in the setting of Hardy or Bergman spaces on Δ (see [9] and [26]).

In the case of an interior null point of a generator, the eigenvalues of Γ_f form a discrete set, while in case of the boundary Denjoy–Wolff point, the spectrum of Γ_f may be larger essentially.

Indeed, as we have already seen in Theorem 3.4, the eigenfunction h corresponding the eigenvalue 2β, where $\beta = f'(1)$, is univalent and starlike with respect to a boundary point. Then $h \in H^p$ for each $p < \frac{1}{2}$ (see, for example, [15]). Hence, for each positive λ the function $h_1(z) := (h(z))^{\frac{2\beta}{\lambda}} \in H^q$ when $q\lambda < \beta$. Obviously, h_1 is an eigenfunction corresponding to λ. Therefore, for any Hardy space H^q we have $\sigma(\Gamma_f) \supset \left(0, \frac{\beta}{q}\right)$.

This, inter alia, motivates consideration of the univalence of eigenfunctions of composition operators on the (locally convex) Fréchet space $\mathrm{Hol}(\Delta, \mathbb{C})$. In a slightly more general setting, we describe the structure of the spectrum of weighted composition operators in the context of the k-valency of the corresponding eigenfunctions. Namely, we use the following definition.

Definition 3.3 (see [16], p. 89) *A function f meromorphic in a domain D is said to be k-valent in D if for each w_0 (infinity included) the equation*

$f(z) = w_0$ has at most k roots in D (where the roots are counted in accordance with their multiplicity) and if there is some w_1 such that the equation $f(z) = w_1$ has exactly k roots in D.

Let E be a space of meromorphic functions in Δ, and let $\mathcal{S} = \{F_t\}_{t\geq 0}$ be a semigroup of holomorphic self-mappings of Δ generated by $f \in \mathcal{G}^+[1],\ f'(1) = \beta$. For a suitable $w \in E$, one can define a (weighted) composition semigroup of linear operators $T_t:\ E \mapsto E,\ t \geq 0$, by the formula

$$T_t(h)(z) = \frac{w(F_t(z))}{w(z)} h(F_t(z)).$$

This semigroup is generated by the operator $\widetilde{\Gamma}_f$ defined by

$$\widetilde{\Gamma}_f h = h'f + hf\frac{w'}{w}, \quad h \in \mathrm{Hol}(\Delta, \mathbb{C}). \tag{3.42}$$

When $w \equiv 1$, this reduces to the unweighted semigroup of composition operators $\{C_t\}_{t\geq 0}$ generated by Γ_f.

It is clear that for each λ in the point spectrum $\sigma_p(\widetilde{\Gamma}_f)$, the eigenspace E_λ corresponding to λ is one-dimensional. For $k \in \mathbb{N}\bigcup\{\infty\}$, denote by $\sigma^{(k)}$ the subset of $\sigma_p(\widetilde{\Gamma}_f)$ such that for each $\lambda \in \sigma^{(k)}$ the function wh is k-valent whenever $h \in E_\lambda$.

Theorem 3.8 *Let $f \in \mathcal{G}^+[1],\ f'(1) = \beta$, and let the operator $\Gamma_f:\ E \mapsto E$ be defined by* (3.42)*. Then the spectrum $\sigma_p\left(\widetilde{\Gamma}_f\right)$ is the whole complex plane $\mathbb{C}$. Moreover,*

$$\sigma_p\left(\widetilde{\Gamma}_f\right) = \bigcup_{k\in\mathbb{N}\cup\{\infty\}} \sigma^{(k)},$$

where for $k \in \mathbb{N}$

$$\sigma^{(k)} = (k\Omega_+ \cup k\Omega_-) \setminus ((k-1)\Omega_+ \cup (k-1)\Omega_-)$$

with $\Omega_\pm = \{\omega \neq 0:\ |\omega \mp \beta| \leq \beta\}$, and

$$\sigma^{(\infty)} = \{\lambda \in \mathbb{C}:\ \mathrm{Re}\,\lambda = 0\}.$$

In addition, for each $\lambda \neq 0$ and for each element h of the eigenspace corresponding to λ, the following modified Robertson inequality holds:

$$\mathrm{Re}\left[\frac{2\beta}{\lambda}\frac{zh'(z)}{h(z)} + \frac{1+z}{1-z} + \frac{2\beta}{\lambda}\frac{zw'(z)}{w(z)}\right] > 0. \tag{3.43}$$

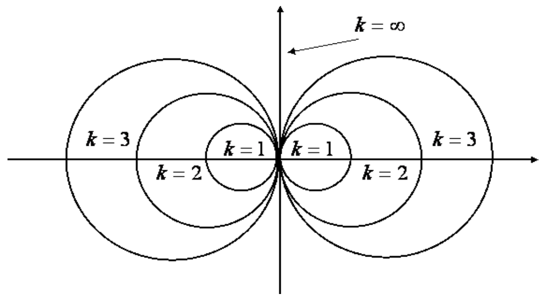

Figure 6: The sets $\sigma^{(k)}$ of k-valence of eigenfunctions.

Fig. 6 illustrates the sets $\sigma^{(k)}$ described in Theorem 3.8.

Proof. The eigenvalue problem for the operator $\widetilde{\Gamma}_f$ is the differential equation

$$h'f + hf\frac{w'}{w} = \lambda h. \tag{3.44}$$

To solve it, we denote by h^* the starlike function with respect to a boundary point which satisfies

$$\beta h^* = h^{*\prime} f \tag{3.45}$$

and is normalized by $h^*(0) = 1$, $h^*(1) = 0$ (h^* is the eigenfunction of Γ_f corresponding to the eigenvalue β).

Then (3.44) and (3.45) imply that

$$\frac{(wh)'}{wh} = \frac{\lambda}{\beta}\frac{h^{*\prime}}{h^*},$$

and hence

$$w(z)h(z) = a\left(h^*(z)\right)^{\lambda/\beta}, \quad a \in \mathbb{C}. \tag{3.46}$$

For a given eigenvalue λ, (3.46) describes the eigenspace corresponding to λ. So it is enough to prove our assertion for the case $w \equiv 1$ or, what is the same, $\widetilde{\Gamma}_f = \Gamma_f$.

To this end, let $\lambda \in \mathbb{C}$. First suppose that $\operatorname{Re}\lambda > 0$ and $\lambda \in k\Omega_+ \setminus (k-1)\Omega_+$. Then $\lambda_1 = \frac{\lambda}{k}$ belongs to Ω_+. Thus, by Theorem 3.4, the solution h_1 of the initial value problem

$$h_1'(z)f(z) = \lambda_1 h_1(z), \quad h_1(0) = 1,$$

is univalent. Set $h(z) = h_1(z)^k$. Evidently, this function satisfies the equation

$$\Gamma_f h(z) \, (= h'(z)f(z)) = \lambda h(z),$$

and is at most k-valent. To show that h is k-valent, consider the univalent solution h_2 of the initial value problem

$$h_2'(z)f(z) = 2\beta h_2(z), \quad h_2(0) = 1.$$

This function is starlike with respect to a boundary point, and the smallest wedge which contains the image $h_2(\Delta)$ is exactly of angle 2π (see the discussion after Definition 3.2). Therefore, for each $\epsilon > 0$, there is real ψ and $r > 0$ small enough such that the image $h_2(\Delta)$ contains the curve

$$\left\{ r \exp\left(\phi \frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\right) \exp\left(i\phi - i \log r \frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\right) : \ \psi \le \phi \le \psi + 2\pi(1-\epsilon) \right\}.$$

Since $\lambda \notin (k-1)\Omega_+$, one can choose $\epsilon < 1 - \frac{2\beta(k-1)\operatorname{Re}\lambda}{|\lambda|^2}$. Then there are k different points $z_0, z_1, \ldots z_{k-1}$ in Δ for which

$$h_2(z_m) = r \exp\left(\phi_m \frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\right) \exp\left(i\phi_m - i \log r \frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\right),$$
$$\phi_m = \psi + 2\pi m \, \frac{2\beta \operatorname{Re}\lambda}{|\lambda|^2}, \quad m = 0, 1, \ldots, k-1.$$

Note that $h(z) = h_2(z)^{\lambda/2\beta}$. Now a simple calculation shows that $h(z_m)$ does not depend on $m = 0, 1, \ldots, k-1$, i.e., h is k-valent and $\lambda \in \sigma^{(k)}$.

The case $\operatorname{Re}\lambda < 0$ is reduced to the previous one by replacing h by $1/h$.

Finally, suppose $\operatorname{Re}\lambda = 0$. As usual, denote by $\{F_t\}_{t\ge 0}$ the semigroup generated by f. Thus, the function $u(t,z) := F_t(z)$ satisfies the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,z)}{\partial t} + f(u(t,z)) = 0 \\ u(0,z) = z \in \Delta. \end{cases}$$

Solving this problem for $z = 0$, we have

$$\int_0^{u(t,0)} \frac{du}{f(u)} = -\int_0^t dt = -t.$$

Since the function $1/f$ is holomorphic on the open disk Δ, we conclude that for each point z which belongs to the curve $\Lambda := \{u(t,0) : t \geq 0\}$ (joining the origin with the boundary point $z = 1$), the integral

$$\int_0^z \frac{du}{f(u)}$$

takes real values which tend to $-\infty$ as $z \in \Lambda$ tends to 1.

Returning to the eigenvalue problem

$$\Gamma_f(h)(z)\,(:= h'(z)f(z)) = \lambda h(z)$$

and separating variables in this differential equation, we have

$$\int_0^z \frac{dh(z)}{h(z)} = \int_0^z \frac{\lambda dz}{f(z)}\,,$$

so that

$$\log\left(\frac{h(z)}{h(0)}\right) = \lambda \int_0^z \frac{dz}{f(z)}\,.$$

Thus, for each point $z \in \Lambda$,

$$h(z) = h(0)\exp\left(-\lambda t\right),$$

where $z = u(t,0)$. We claim that this function is infinite-valent. When $\lambda = 0$, this is evident. Suppose $\lambda \neq 0$; then $\lambda = ia,\ a \neq 0$. Choosing the sequence $z_n = u\left(t_0 + \dfrac{2\pi n}{a}, 0\right) \in \Lambda \subset \Delta$ for any fixed $t_0 \geq 0$, we see immediately that the value $h(z_n)$ does not depend on n.

Now, let h^* be the solution of the differential equation (3.45) normalized by $h^*(0) = 1,\ h^*(1) = 0$. As in the proof of Theorem 3.3, we conclude that the image of the function $h^*(\Delta)$ must lie in a wedge of angle π. Since h^* is starlike with respect to a boundary point, it follows from a result of Lyzzaik [19] (see also [24]) that

$$\mathrm{Re}\left[\frac{2zh^{*\prime}(z)}{h^*} + \frac{1+z}{1-z}\right] > 0.$$

Substituting (3.46) into the latter inequality, we get (3.43). The proof is complete.

Combining this assertion with previous results, one can obtain additional geometrical information on (characteristics of) eigenfunctions.

Example 4. For each $a \in (0,1)$, we rewrite (3.43) in the form

$$\operatorname{Re}\left[\frac{2a\beta}{\lambda}\,\frac{zh'(z)}{h(z)} + \frac{1+z}{1-z}\right] > \operatorname{Re}\left[(1-a)\,\frac{1+z}{1-z} - \frac{2a\beta}{\lambda}\,\frac{zw'(z)}{w(z)}\right]. \tag{3.47}$$

Suppose that $w(z) = (1-z)^{\delta}$ for some $\delta \in \mathbb{C}$.

Let c be a positive number less than 1. In this case, for each λ such that $\frac{\delta}{\lambda}$ is real with $\frac{\delta}{\lambda} < -\frac{c}{a\beta}$, a simple calculation shows that the right-hand side in (3.47) is greater than c. Indeed, for each $z \in \partial\Delta$,

$$\begin{aligned}\operatorname{Re}\left[(1-a)\,\frac{1+z}{1-z} - \frac{2a\beta}{\lambda}\,\frac{zw'(z)}{w(z)}\right] &= \frac{1}{|1-z|^2}\operatorname{Re}\left[-\frac{2a\beta\delta}{\lambda} + \frac{2a\beta\delta z}{\lambda}\right] \\ &= -\frac{2a\beta\delta}{\lambda}\operatorname{Re}\frac{1-z}{|1-z|^2} \ge -\frac{a\beta\delta}{\lambda} > c.\end{aligned}$$

Letting $a \to 1^-$, we conclude that

$$\operatorname{Re}\left[\frac{2\beta}{\lambda}\,\frac{zh'(z)}{h(z)} + \frac{1+z}{1-z}\right] > c.$$

Therefore, by Theorem 3.1 in [10], the image $h(\Delta)$ of the eigenfunction h covers the image of the function $(1-z)^{\frac{c\lambda}{\beta}}$.

Example 5. Consider now the (nonanalytic) weight $w(z) = \frac{(1-z)^2}{z}$. By formula (3.46), each eigenfunction has the form

$$h(z) = a\,\frac{z}{(1-z)^2}\,(h^*(z))^{\frac{\lambda}{\beta}}.$$

In this situation, for all positive λ less than 2β, inequality (3.43) becomes

$$\operatorname{Re}\frac{zh'(z)}{h(z)} > \left(1 - \frac{\lambda}{2\beta}\right)\frac{1-|z|^2}{|1-z|^2} > 0,$$

i.e., all eigenfunctions are starlike with respect to an interior point ($h(0) = 0$).

In particular, the function

$$h_0(z) = \frac{z}{(1-z)^2}\,h^*(z)$$

is the eigenfunction corresponding the eigenvalue $\lambda = \beta$. Since $h^* \in \mathrm{Sp}_1[1]$, Theorem 3.6 implies that h_0 is a starlike function of class $S^*[0]$ which satisfies the condition

$$2 \inf \left(\frac{1 - |z|^2}{|1 - z|^2} \operatorname{Re} \frac{z h_0'(z)}{h_0(z)} \right) = 1.$$

Finally, we observe that the above theorem implies the following nice characterization of multivalent starlike functions with respect to a boundary point suggested by D. Bshouty and A. Lyzzaik (see [7] and [8]; cf. also [17] and [18]).

Theorem 3.9 *Let $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ satisfy the equation*

$$-(1 - z)^2 \frac{h'(z)}{h(z)} = 4 \frac{1 - \omega(z)}{1 + \omega(z)}, \tag{3.48}$$

where $\omega \in \mathrm{Hol}(\Delta)$ is a holomorphic self-mapping of Δ with the boundary fixed point $\tau = 1$ and positive multiplier

$$\alpha = \angle \lim_{z \to 1} \frac{1 - \omega(z)}{1 - z}.$$

Then

(i) $h(\Delta)$ is a starlike domain;

(ii) h is a k-valent function if and only if $k - 1 < \alpha \le k$.

In particular, if $\tau = 1$ is the Denjoy–Wolff point for ω (whence $0 < \alpha \le 1$), then h is univalent and the smallest wedge which contains $h(\Delta)$ is of angle $2\alpha\pi$.

Proof. Let

$$p(z) = \frac{\alpha}{2} \frac{1 + \omega(z)}{1 - \omega(z)}.$$

This function has positive real part. Moreover,

$$\angle \lim_{z \to 1} (1 - z) p(z) = \angle \lim_{z \to 1} \frac{\alpha}{2} \frac{1 - z}{1 - \omega(z)} (1 + \omega(z)) = 1$$

Define a generator $f \in \mathcal{G}(\Delta)$ by the Berkson–Porta formula

$$f(z) = -(1 - z)^2 p(z).$$

It is clear that $f \in \mathcal{G}^+[1]$ with $\beta = f'(1) = 1$.

Since h satisfies the equation

$$2\alpha h(z) = h'(z)f(z),$$

and α is positive, applying Theorem 3.8 with $\lambda = 2\alpha$ and $\beta = 1$, we obtain the result.

If $\tau = 1$ is the Denjoy–Wolff point for ω, i.e., $0 < \alpha \le 1$, then

$$\angle \lim_{z \to 1} \frac{(z-1)h'(z)}{h(z)} = 2\alpha \le 2;$$

and the assertion follows by Theorem 7 in [12]. The proof is complete.

References

[1] M. Abate, The infinitesimal generators of semigroups of holomorphic maps, *Ann. Mat. Pura Appl.* (4) **161** (1992), 167–180.

[2] D. Aharonov, M. Elin, S. Reich and D. Shoikhet, Parametric representations of semi-complete vector fields on the unit balls in $\mathbb{C}^n$ and Hilbert space, *Atti Accad. Naz. Lincei Cl. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **10** (1999), 229–253.

[3] D. Aharonov, M. Elin and D. Shoikhet, Spiral-like functions with respect to a boundary point, *J. Math. Anal. Appl.* **280** (2003), 17–29.

[4] D. Aharonov, S. Reich and D. Shoikhet, Flow invariance conditions for holomorphic mappings in Banach spaces, *Math. Proc. R. Ir. Acad.* **99A** (1999), 93–104.

[5] J. Arazy, An application of infinite dimensional holomorphy to the geometry of Banach spaces, *Geometrical Aspects of Functional Analysis (1985/86)*, Springer Lecture Notes in Math. **1267**, 1987, 122–150.

[6] E. Berkson and H. Porta, Semigroups of analytic functions and composition operators, *Michigan Math. J.* **25** (1978), 101–115.

[7] D. Bshouty and A. Lyzzaik, Univalent functions starlike with respect to a boundary point, *Complex Analysis and Dynamical Systems II, Contemporary Mathematics* **382** (2005),

[8] D. Bshouty and A. Lyzzaik, Personal communication.

[9] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[10] M. Elin, Covering and distortion theorems for spirallike functions with respect to a boundary point, Preprint, 2005.

[11] M. Elin, S. Reich and D. Shoikhet, Holomorphically accretive mappings and spiral-shaped functions of proper contractions, *Nonlinear Anal. Forum* **5** (2000), 149–161.

[12] M. Elin, S. Reich and D. Shoikhet, Dynamics of inequalities in geometric function theory, *J. Inequal. Appl.* **6** (2001), 651–664.

[13] M. Elin, S. Reich and D. Shoikhet, Asymptotic behavior of semigroups of ρ-nonexpansive and holomorphic mappings on the Hilbert ball, *Ann. Mat. Pura Appl.* (4) **181** (2002), 501–526.

[14] M. Elin and D. Shoikhet, Dynamic extension of the Julia–Wolff–Carathéodory theorem, *Dynam. Systems Appl.* **10** (2001), 421–437.

[15] J. B. Garnett, *Bounded Analytic functions*, Academic Press, 1981.

[16] A. W. Goodman, *Univalent Functions*, Vols. I, II, Mariner Publ. Co., Tampa, FL, 1983.

[17] A. Lecko, On the class of functions starlike with respect to a boundary point, *J. Math. Anal. Appl.* **261** (2001), 649–664.

[18] A. Lecko and A. Lyzzaik, A note on univalent functions starlike with respect to a boundary point, *J. Math. Anal. Appl.* **282** (2003), 846–851.

[19] A. Lyzzaik, On a conjecture of M.S. Robertson, *Proc. Amer. Math. Soc.* **91** (1984), 108–110.

[20] Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer, Berlin, 1992.

[21] S. Reich and D. Shoikhet, Generation theory for semigroups of holomorphic mappings in Banach spaces, *Abstr. Appl. Anal.* **1** (1996), 1–44.

[22] S. Reich and D. Shoikhet, The Denjoy–Wolff theorem, *Ann. Univ. Mariae Curie–Sklodowska Sect. A* **51** (1997), 219–240.

[23] S. Reich and D. Shoikhet, Metric domains, holomorphic mappings and nonlinear semigroups, *Abstr. Appl. Anal.* **3** (1998), 203–228.

[24] M. S. Robertson, Univalent functions starlike with respect to a boundary point, *J. Math. Anal. Appl.* **81** (1981), 327–345.

[25] W. Rudin, *Function Theory on the Unit Ball in* $\mathbb{C}^n$, Springer, Berlin, 1980.

[26] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, Berlin, 1993.

[27] D. Shoikhet, *Semigroups in Geometrical Function Theory*, Kluwer, Dordrecht, 2001.

[28] D. Shoikhet, Representations of holomorphic generators and distortion theorems for spirallike functions with respect to a boundary point, *Int. J. Pure Appl. Math.* **5** (2003), 335–361.

[29] H. Silverman and E. M. Silvia, Subclasses of univalent functions starlike with respect to a boundary point, *Houston J. Math.* **16** (1990), 289–299.

[30] H. Upmeier, *Jordan Algebras in Analysis, Operator Theory and Quantum Mechanics*, Amer. Math. Soc., Providence, RI, 1987.

[31] K. Yosida, *Functional Analysis*, Springer, Berlin, 1968.